

THE MAXIMUM PRINCIPLE FOR MINIMAL VARIETIES OF ARBITRARY CODIMENSION

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ABSTRACT. We prove that an m -dimensional minimal variety in a Riemannian manifold cannot touch the boundary at a point where the sum of the smallest m principal curvatures is greater than 0. We also prove an analogous result for varieties with bounded mean curvature.

Let N be a smooth Riemannian manifold with boundary. In general, N need not be complete. Suppose X is a compactly supported C^1 tangent vectorfield on N such that

$$(1) \quad X \cdot \nu_N \geq 0$$

at all points of ∂N , where ν_N is the unit normal to ∂N that points into N . Then X generates a one-parameter family $t \in [0, \infty) \mapsto \phi_t$ of maps of N into itself such that ϕ_0 is the identity map and such that

$$\frac{d}{dt} \phi_t(\cdot) = X(\phi_t(\cdot)).$$

If V is a C^1 submanifold of N with finite area, we let $\delta V(X)$ denote the first variation of area of V with respect to X :

$$\delta V(X) = \left(\frac{d}{dt} \right)_{(t=0)} \text{area}(\phi_t(V)).$$

More generally, if V has locally finite area, we let

$$\delta V(X) = \left(\frac{d}{dt} \right)_{(t=0)} \text{area}(\phi_t(V \cap W))$$

where W is any open subset of N that contains the support of X and that has compact closure. Even more generally, V can be any varifold in N . (The theorems in this paper are stated for arbitrary varifolds, but readers unfamiliar with varifolds may substitute “ C^1 submanifold” for “varifold” throughout the paper. The appendix to [Whi09] is a brief introduction to varifolds that contains most of what is needed here. For a more thorough treatment, see [Sim83] or [All72].)

We say that a varifold V in N *minimizes area to first order* in N provided

$$\delta V(X) \geq 0$$

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for every compactly supported C^1 tangent vectorfield X on N satisfying (1). In particular, any smooth minimal submanifold of N or, more generally, any stationary varifold in N minimizes area to first order in N .

We say that N is *strongly m -convex* at a point $p \in \partial N$ provided

$$\kappa_1 + \kappa_2 + \cdots + \kappa_m > 0$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$ are the principal curvatures of ∂N at p with respect to the unit normal ν_N that points into N . We say that N is m -convex at p provided $\kappa_1 + \kappa_2 + \cdots + \kappa_m \geq 0$.

Theorem 1 (Maximum Principle for Minimal Varifolds). *Let N be a smooth Riemannian manifold with boundary, and let p be a point in ∂N at which N is strongly m -convex. Then p is not contained in the support of any m -dimensional varifold in N that minimizes area to first order in N . Indeed, there is an $\epsilon > 0$ such that*

$$\text{dist}(p, \text{spt } V) \geq \epsilon$$

for all such varifolds V .

In case V is a smooth minimal submanifold, the fact that V cannot contain p was proved by Jorge and Tomi [JT03]. Indeed, they proved that if N is m -convex (not necessarily strongly m -convex) at all boundary points, then a smooth, connected minimal m -dimensional submanifold cannot touch ∂N unless it lies entirely in ∂N . An analogous result for smooth submanifolds V of bounded mean curvature was proved by Dierkes and Schwab [DS05] assuming (in addition to the appropriate condition on ∂N) that N is flat. The proofs here are similar to the proofs in those papers. The results here are stronger in that they apply to nonsmooth varieties (varifolds), and in that the varieties are not assumed to be stationary with respect to all variations, but only with respect to variations that take N into itself (i.e., that satisfy (1)). This extra generality is important because when one solves variational problems for area in a manifold-with-boundary N , the solution surfaces need not be stationary (or minimal), but rather only stationary with respect to variations satisfying (1). Likewise, even if a solution surface turns out to be a differentiable manifold, it need not be smooth – it may only be $C^{1,1}$. Of course if the boundary of N is strictly m -convex, then according to Theorem 1, the solution surface does lie in the interior of N , which implies that it is stationary with respect to all variations, and that if it is a differentiable manifold, then it is a smooth submanifold. But it is important that these properties of the surface are conclusions of Theorem 1 rather than hypotheses.

Theorem 1 is proved by constructing a suitable test vectorfield X :

Theorem 2. *Let N be a smooth Riemannian manifold with boundary. Let p be a point in ∂N and let $\eta < \kappa_1 + \cdots + \kappa_m$, where $\kappa_1 \leq \cdots \leq \kappa_{n-1}$ are the principal curvatures of ∂N at p with respect to the unit normal ν_N that points into N . Then there is a compactly supported C^∞ vectorfield X on N such that $X(p)$ is a nonzero normal to ∂N , such that*

$$(2) \quad X \cdot \nu_N \geq 0 \text{ at all points of } \partial N,$$

and such that

$$(3) \quad \delta V(X) \leq -\eta \int |X| d\mu_V$$

for every m -dimensional varifold V in N .

Here μ_V is the weight measure associated to V . (If V is a C^1 submanifold, then the integration in (3) is simply integration over V with respect to m -dimensional area.)

We remark that X can be chosen so that its support is contained in an arbitrarily small neighborhood of p and so that the vectorfield $X/|X|$ (wherever X is nonzero) is arbitrarily C^0 -close to $\nabla \text{dist}(\cdot, \partial N)$. (In the proof below, one simply chooses ϵ sufficiently small.)

To see that Theorem 1 follows from Theorem 2, note that if N is strongly m -convex at p , then we may choose the η in Theorem 2 to be positive. If V minimizes area to first order in N , then by definition and by Theorem 2,

$$0 \leq \delta V(X) \leq -\eta \int |X| d\mu_V.$$

Since $\eta > 0$, this implies that $|X|$ vanishes μ_V -almost everywhere and thus that the support of V cannot contain any point where $X \neq 0$. Hence $\text{dist}(p, \text{spt } V) \geq \epsilon$, where ϵ is the distance from p to the nearest point where X vanishes.

Proof of Theorem 2. Given a compactly supported C^1 vectorfield X on N , let $\Psi_X : N \rightarrow \mathbf{R}$ be the function

$$\Psi_X(x) = \max (\text{trace}(\nabla X|P))$$

where the maximum is over all m -dimensional linear subspaces P of $\text{Tan}_x N$ and where

$$\text{trace}(\nabla X|P) = \sum_{i=1}^m \mathbf{u}_i \cdot \nabla_{\mathbf{u}_i} X$$

for any orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_m$ of P .

If V is an m -dimensional C^1 submanifold of N , then by the first variation formula [Sim83, §9.3]¹

$$\begin{aligned} (4) \quad \delta V(X) &= \int \text{trace}(\nabla X| \text{Tan}_x V) d\mu_V x. \\ &\leq \int \Psi_X d\mu_V. \end{aligned}$$

More generally, if V is any m -dimensional varifold in N , then by the first variation formula [Sim83, §39.2],

$$\begin{aligned} \delta V(X) &= \int_{(x,P) \in G_m(N)} \text{trace}(\nabla X|P) dV(x,P) \\ &\leq \int_{(x,P) \in G_m(N)} \Psi_X(x) dV(x,P) \\ &= \int \Psi_X d\mu_V. \end{aligned}$$

where $G_m(N)$ is the set of pairs (x, P) such that $x \in N$ and P is an m -dimensional linear subspace of $\text{Tan}_x N$.

Thus we see that the conclusion (3) of Theorem 2 will hold provided

$$(5) \quad \Psi_X(\cdot) \leq -\eta |X(\cdot)|$$

¹ Equation (4) is proved by expressing the area of $\phi_t(V)$ as the integral of a Jacobian determinant and then differentiating under the integral sign. If V is a smooth submanifold, one can then integrate by parts to express $\delta V(X)$ as $-\int X \cdot H d\mu_V$.

at all points of N .

To construct the desired vectorfield X , we may assume that N is part of a larger Riemannian manifold \tilde{N} (without boundary) of the same dimension². Let

$$\Sigma = \{q \in \tilde{N} : \text{dist}(x, N) = \text{dist}(x, p)^4\}.$$

Note that Σ and ∂N make second order contact at p . By replacing \tilde{N} with a small geodesic ball around p , we may assume that Σ is a smooth hypersurface and that there is smooth, well-defined nearest-point retraction from \tilde{N} to Σ . (We will later replace \tilde{N} by an even smaller ball to ensure that additional conditions are satisfied.)

For $x \in \tilde{N}$, let $u(x)$ be the signed distance from x to Σ , with the sign chosen so that u is nonnegative on N . For $q \in \tilde{N}$, let Σ_q be the level set of u that contains q . Note that $\nu(q) := \nabla u(q)$ is a unit normal to Σ_q . Let

$$(6) \quad k_1(q) \leq \cdots \leq k_{n-1}(q)$$

be the principal curvatures of Σ_q at q with respect to the unit normal $\nu(q)$.

Note that

$$(7) \quad k_1 + \cdots + k_m > \eta$$

at p since Σ and ∂N make second order contact at p . By replacing \tilde{N} with a sufficiently small ball around p , we may assume that (7) holds at all points of \tilde{N} , that

$$(8) \quad \nu \cdot \nu_N > 0 \text{ at all points of } \partial N,$$

and that the $|k_i|$ are uniformly bounded:

$$(9) \quad |k_i(q)| \leq K \quad (q \in \tilde{N}, i \leq n-1).$$

Let ϵ be a positive number (to be specified later), and define a vectorfield X on N by

$$X(\cdot) = \phi(u(\cdot)) \nu(\cdot)$$

where

$$\phi(t) = \begin{cases} \exp\left(\frac{1}{t-\epsilon}\right) & \text{if } 0 \leq t < \epsilon, \\ 0 & \text{if } t \geq \epsilon. \end{cases}$$

(We need not define $\phi(t)$ for $t < 0$ since $u \geq 0$ on N .) Note that

$$\frac{\phi'(t)}{\phi(t)} = \frac{-1}{(t-\epsilon)^2} \leq \frac{-1}{\epsilon^2}$$

for $0 \leq t < \epsilon$, and thus

$$\phi'(t) \leq -\frac{1}{\epsilon^2} \phi(t)$$

for all $t \geq 0$. Thus by choosing $\epsilon \leq K^{-2}$, we can ensure that

$$(10) \quad \phi'(t) \leq -K\phi(t)$$

for all $t \geq 0$.

We also choose ϵ small enough that $N \cap \{u \leq \epsilon\}$ is compact.

² If the existence of a such a \tilde{N} is not clear, note that p has a neighborhood diffeomorphic to a closed half-space in \mathbf{R}^n . Since Theorem 2 is local, we can assume that N is that half-space with some smooth Riemannian metric. We can extend the Riemannian metric to all of \mathbf{R}^n and then let \tilde{N} be \mathbf{R}^n with the extended metric.

We claim that the vectorfield X has the desired properties. First note that

$$\text{spt } X = \overline{N \cap \{u < \epsilon\}},$$

which is compact by choice of ϵ . Also,

$$X \cdot \nu_N = \phi(u) \nu \cdot \nu_N \geq 0$$

at all points of ∂N by (8), since ϕ is everywhere nonnegative.

It remains only to show that $\Psi_X \leq -\eta |X|$. Let q be any point in N . Let $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ be principal directions in $\text{Tan}_q \Sigma_q$ corresponding to the principal curvatures $k_1(q), \dots, k_{n-1}(q)$. Consider the bilinear form Q on $\text{Tan}_q N$ given by

$$Q(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \nabla_{\mathbf{v}} X.$$

We wish to calculate the matrix for Q with respect to the orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \nu$.

Note that if \mathbf{v} is tangent to Σ_q , then

$$(11) \quad \nabla_{\mathbf{v}} X = \nabla_{\mathbf{v}}(\phi(u)\nu) = \phi(u)\nabla_{\mathbf{v}}\nu$$

since $\phi(u)$ is constant on Σ_q . Thus if \mathbf{u} and \mathbf{v} are both tangent to Σ_q , then

$$Q(\mathbf{u}, \mathbf{v}) = \phi(u)\mathbf{u} \cdot \nabla_{\mathbf{v}}\nu = -\phi(u)B(\mathbf{u}, \mathbf{v})$$

where B is the second fundamental form of Σ_q with respect to the normal ν . In particular,

$$Q(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} -\phi(u)k_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $\|\nu\| \equiv 1$, we see that $\nabla_{\mathbf{v}}\nu$ is perpendicular to ν and thus

$$Q(\nu, \mathbf{e}_i) = 0$$

by (11). Since ν is the gradient of the distance function, $\nabla_{\nu}\nu = 0$. Thus

$$\begin{aligned} \nabla_{\nu} X &= \nabla_{\nu}(\phi(u)\nu) \\ &= \phi'(u)(\nabla_{\nu}u)\nu + \phi(u)\nabla_{\nu}\nu \\ &= \phi'(u)\nu, \end{aligned}$$

so $Q(\nu, \nu) = \phi'(u)$ and $Q(\mathbf{e}_i, \nu) = 0$.

Hence we see that the matrix for Q with respect to the orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \nu$ is a diagonal matrix with diagonal elements $-\phi(u)k_i$ (for $1 \leq i \leq n-1$) and $\phi'(u)$. Note that

$$-\phi(u)k_1 \geq -\phi(u)k_2 \geq \dots \geq -\phi(u)k_{n-1} \geq -\phi(u)K \geq \phi'(u)$$

by (6), (9), and (10), since $\phi \geq 0$. In particular, since $\phi(u) = |X|$, the largest m eigenvalues of Q are $-|X(q)|k_i$ where $1 \leq i \leq m$. It follows by elementary linear algebra that

$$\begin{aligned} \Psi(q) &= \max_P \text{trace}(Q|P) \\ &= -|X(q)| (k_1(q) + \dots + k_m(q)) \\ &\leq -\eta |X(q)| \end{aligned}$$

by (7). This completes the proof. \square

Theorem 3 (Maximum Principle for Set-Theoretic Limits of Minimal Varieties). *Suppose N_i is a sequence of smooth Riemannian n -manifolds with boundary, and suppose that the N_i converge smoothly to a limit Riemannian manifold N . Suppose for each i that V_i is an m -dimensional varifold in N_i that minimizes area to first order in N_i , and suppose that the sets $\text{spt}(V_i)$ converge to a limit set $S \subset N$. Then S does not contain any point of ∂N at which N is strongly m -convex.*

Proof. Since the result is local, we may assume that the N_i and N are all the same as smooth manifolds but have Riemannian metrics $g(i)$ and g where $g(i)$ converges smoothly to g .

Let $\kappa_1(\cdot) \leq \dots \leq \kappa_{n-1}(\cdot)$ be the principal curvatures of ∂N with respect to the inward pointing unit normal.

Let p be a point of ∂N at which N is strongly m -convex (with respect to g .) Let $0 < \eta < \kappa_1(p) + \dots + \kappa_m(p)$. In the proof of Theorem 2, we constructed a smooth function $u : N \rightarrow \mathbf{R}$ with the following properties (with respect to the metric g):

- (i) $u(p) = 0$ and $u > 0$ on $N \setminus \{p\}$.
- (ii) The set $C = \{u \leq \epsilon\}$ is compact.
- (iii) $\kappa_1 + \dots + \kappa_m > \eta$ at all points of $C \cap \partial N$.
- (iv) ∇u never vanishes on C , and

$$k_1(q) + \dots + k_m(q) > \eta$$

at each point $q \in C$, where $k_1(q) \leq \dots \leq k_{n-1}(q)$ are the principal curvatures of the level set $\Sigma_q = \{x : u(x) = u(q)\}$ with respect to the unit normal $\nabla u(q)$.

By the smooth convergence $g(i) \rightarrow g$, these properties will also hold with respect to the metric $g(i)$ for all sufficiently large i . Fix such an i . We claim that $\text{spt } V_i$ cannot contain any point of C . For if it did, the function u restricted to $C \cap \text{spt } V_i$ would attain a minimum at some point q . By (iii) and by Theorem 1, q cannot be in ∂N . By (iv), the set $\{u \geq u(q)\}$ is strongly m -convex at q , which contradicts Theorem 1 (since $q \in \text{spt } V_i \subset \{u \geq u(q)\}$.) Thus $C \cap \text{spt } V_i$ is empty. Since p is in the interior of C , we are done. \square

In the case of hypersurfaces (i.e., when $m = \dim(N) - 1$), Solomon and White proved the following strong maximum principle for minimal varieties [SW89]:

Theorem 4. *Suppose that N is a smooth Riemannian manifold with boundary and that N is mean convex on some connected open subset U of ∂N , i.e., that*

$$H \cdot \nu_N \geq 0$$

on U , where H is the mean curvature vector of ∂N and where ν_N is the unit normal to ∂N that points into N . Let $m = \dim(N) - 1$, and suppose that V is an m -dimensional varifold that minimizes area to first order in N . If $\text{spt } V$ contains any point of U , then it must contain all of U and H must vanish everywhere on U .

Theorem 4 is actually a consequence of Theorem 1: see [SW89, Step 1, p. 687] and the comments at the end of [SW89].

See [Ilm96] and [Sch04] for other strong maximum principles for varieties of codimension 1. In particular, [Ilm96] gives a very general strong maximum principle for pairs of codimension 1 minimal varieties, both of which may be singular.

Theorem 5 (Maximum Principle for Varieties with Bounded Mean Curvature). *Let N be a smooth Riemannian manifold with boundary and h be a nonnegative number. Suppose V is an m -dimensional varifold in N and that*

$$(12) \quad \delta V(X) + h \int |X| d\mu_V \geq 0$$

for every compactly supported C^1 vectorfield on N such that

$$(13) \quad X \cdot \nu_N \geq 0 \text{ at all points of } \partial N.$$

Then the support of V cannot contain any point p in ∂N at which

$$\kappa_1 + \cdots + \kappa_m > h$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$ are the principal curvatures of ∂N with respect to the unit normal ν_N that points into N .

Indeed, there is an $\epsilon = \epsilon(h)$ such that $\text{dist}(p, \text{spt } V) \geq \epsilon$ for all m -varifolds V satisfying (12).

To understand the meaning of the hypothesis on V , suppose that V is a smooth m -dimensional submanifold. In that case, the inequality (12) holds for all compactly supported C^1 vectorfields if and only if the length of the mean curvature vector of V is everywhere bounded by h . The inequality holds for all X satisfying (13) if and only if: (i) at every point in $V \setminus \partial N$, the length of the mean curvature vector is at most h , and (ii) at every point $q \in V \cap \partial N$, the mean curvature vector at q can be written as the sum of a vector of length at most h and a normal vector to ∂N that points out of N .

Theorem 5 follows from Theorem 2 exactly as Theorem 1 did. (One chooses the η in Theorem 2 to be strictly between h and $\kappa_1(p) + \cdots + \kappa_m(p)$.) Note that Theorem 1 is Theorem 5 in the special case $h = 0$.

Theorem 6 (Maximum Principle for Set-Theoretic Limits of Varieties with Bounded Mean Curvature). *Suppose that N_i is a sequence of smooth Riemannian n -manifolds with boundary, and suppose that the N_i converge smoothly to a limit Riemannian manifold N . Suppose for each i that V_i is an m -dimensional varifold in N_i and that*

$$\delta V_i(X) + h \int |X| d\mu_{V_i} \geq 0$$

for every compactly supported C^1 vectorfield X on N_i such that

$$X \cdot \nu_{N_i} \geq 0 \text{ at all points of } \partial N_i.$$

Suppose also that the sets $\text{spt}(V_i)$ converge to a limit set $S \subset N$. Then S does not contain any point of ∂N at which

$$\kappa_1 + \cdots + \kappa_m > h,$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$ are the principal curvatures of ∂N with respect to the unit normal that points into N .

The proof is almost identical to the proof of Theorem 3.

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